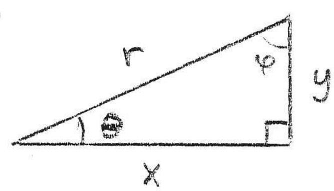
Right Triangle Approach to Trig: **The Three "Unit" Right Triangles**

▪ Standard Position & Labelling of Sides and Angles

For consistency and clarity in our discussions, we'll often draw our right triangles in "**standard position**": meaning the first leg is drawn horizontally left-to-right; the second leg is drawn vertically upward; and the hypotenuese is drawn making an acute angle at the left vertex of the horizontal side. (See diagram below.)

Note that a right triangle with any initial positioning can be rotated (and perhaps flipped) into the "standard position" described above. Rotations and flips (= reflections) do not change *lengths of sides* or *angles between sides* of a triangle. So we incur no loss of generality in our study of relationships between the side lengths and (acute) angles of a triangle by working with triangles in standard position.

Let's label the horizontal side length x, the vertical side length y, the hypotenuese length r. Let's also label the angle at the bottom-left by the lower case greek letter, θ (pronounced "theta"), and the angle at the upper-right by φ (pronounced "phi"). Note that x, y, and r are postive here and θ and φ are necessarily acute (i.e. between 0° and 90°).



Note also that:

▪ Basic Relationships

- With our labelling above, the Pythagorean Theorem tells us: $x^2 + y^2 = r^2$
- From the Angle-Sum Theorem the two acute angles are complementary: $\theta + \varphi = 90^\circ$.

▪ Definitions of the Six Trig Functions for an Input Angle θ with $0^\circ < \theta < 90^\circ$

Recall there are exactly six ways one can form a ratio by using two of the side lengths x, y, and r. And, from the basic facts of similar triangles, those six ratios are unchanged if we dilate or contract each side of a given triangle by any fixed scaling factor. Each of these six ratios are given a name, in reference to the acute angle θ, and can be considered to be functions of θ.

$$\sin(\theta) = \frac{\text{side opposite } \theta}{\text{hypoteneuse}} = \frac{y}{r} \qquad \cos(\theta) = \frac{\text{side adjacent } \theta}{\text{hypoteneuse}} = \frac{x}{r} \qquad \tan(\theta) = \frac{\text{side opposite } \theta}{\text{side adjacent } \theta} = \frac{y}{x}$$

$$\csc(\theta) = \frac{\text{hypoteneuse}}{\text{side opposite } \theta} = \frac{r}{y} \qquad \sec(\theta) = \frac{\text{hypoteneuse}}{\text{side adjacent } \theta} = \frac{r}{x} \qquad \cot(\theta) = \frac{\text{side adjacent } \theta}{\text{side opposite } \theta} = \frac{x}{y}$$

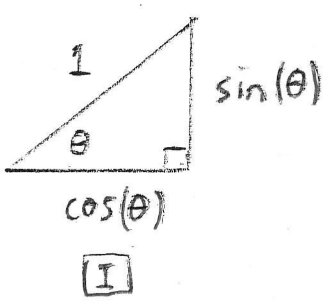
▪ Three Special Scalings of a Right Triangle with Acute Angle θ

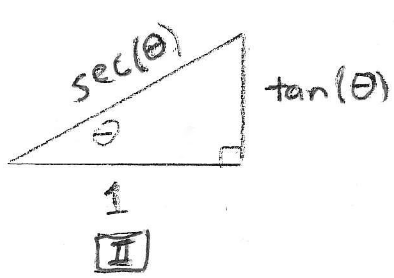
Here's a simple fact: A ratio with 1 in its *denominator* is *equal* to its *numerator*. For any given right triangle with angle θ, we can scale its sides by a common amount to obtain a new similar triangle having one of its sides of unit length. This gives a concrete way to visualize the output values of the six trig functions as side lengths of appropriately scaled triangles. For instance:

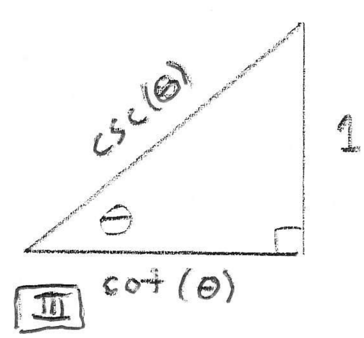
I If we scale our triangle so the hypoteneuse has length $r = 1$, then $\cos(\theta) = x$ (the horizontal side) and $\sin(\theta) = y$ (the vertical side).

II If we scale our triangle so the horizontal leg has length $x = 1$, then $\tan(\theta) = y$ (the vertical side) and $\sec(\theta) = r$ (the hypoteneuse).

III If we scale our triangle so the vertical leg has length $y = 1$, then $\cot(\theta) = x$ (the horizontal side) and $\csc(\theta) = r$ (the hypoteneuse).







Exercises

For each of the triangle below do the following:

- Use the Pythagorean Theorem to find the unknown (i.e. x, y, or r).
- Each triangle has one side of unit length, thus the other two sides correspond to trig function values. Indicate which trig function of θ corresponds to each (non-unit) side length.
- The side legnths in the triangles below have been drawn accurately. Use a protractor to measure the (lower left) angle in the triangles. Then use a calculator to evaluate the trig function values corresponding to the (non-unit length) sides. Compare these with the decimal form of the side lengths you've already computed above. (They should be equal.)
- Use the appropriate inverse trig function buttons on your calculator to find the angle θ from the side lengths you computed earlier. (The angle here should agree with what you measured with your protractor.)