## Using the "ac Method" to Factor $2^{\text {nd }}$ Degree Polynomials

A $2^{\text {nd }}$ degree (or "quadratic") polynomial in the variable $x$ can be written in the following standard form:

$$
a x^{2}+b x+c
$$

Notice that the above expression is a sum of three terms with coefficients $a, b$, and $c$. To factor this means to rewrite it as a product of two $1^{\text {st }}$ degree polynomials...that is, in the form:

$$
(u x+r)(v x+s)
$$

Here all our coefficients $a, b, c, u, r, v$, and $s$ are assumed to be integers, and we say that we are trying to "factor over the integers". Depending on $a$, $b$, and $c$, the polynomial $a x^{2}+b x+c$ may, or may not, factor over the integers. Below we describe a way to determine whether a second degree polynomial factors over the integers (called the "ac test"). And, if it does factor, we give a way (called the "ac method") to find the factorization.

## The "ac Test"

To determine whether a quadratic polynomial $a x^{2}+b x+c$ factors over the integers:

1. First compute the value of the product, $a c$.
2. Next try to find for two integers, call them $p$ and $q$, satisfying the conditions: $p q=a c$ and $p+q=b$.
3. If such a $p$ and $q$ exist. then your quadratic polynomial will factor over the integers. If no such $p$ and $q$ exists, your polynomial will not factor over the integers.
If your polynomial does turn out to factor, we say it "passes the ac test". In that case, you can find the factorization by applying the "ac Method".

## The "ac Method"

To factor a quadratic polynomial $a x^{2}+b x+c$ over the integers:

1. Using the values of $p$ and $q$ found in the "ac Test", rewrite $a x^{2}+b x+c$ in the form $a x^{2}+p x+q x+c$.
2. The expression $a x^{2}+p x+q x+c$ can now be factored by grouping to obtain the factorization you're after.

## Examples

Example 1. Factor $3 x^{2}+8 x+4$, if possible.
Solution: First let's use the "ac test" to see if it does indeed factor. Note that: $a=3, b=8$, and $c=4$. So $a c=3 \cdot 4=12$. Thus we want to search for two integers $p$ and $q$ satisfying:

$$
p q=12 \quad \text { and } \quad p+q=8
$$

If we can quickly "see" a $p$ and $q$ that work, great! But, if not, we can be systematic and make a table (shown below) of all the pairs of factors of $a c=12$, and then check to see if any of those add to 8 . Indeed, from the table below, we see the red row "works"...that is $p=2$ and $q=6$.

| p | q | $\mathrm{p}+\mathrm{q}$ |
| ---: | ---: | ---: |
| 1 | 12 | 13 |
| 2 | 6 | 8 |
| 3 | 4 | 7 |
| -1 | -12 | -13 |
| -2 | -6 | -8 |
| -3 | -4 | -7 |

Thus the polynomial $3 x^{2}+8 x+4$ passes the ac test, so we are guaranteed that it will factor over the integers.
So now, to obtain the factorization of $3 x^{2}+8 x+4$, we use the "ac method". That is, we rewrite the $8 x$ term as $2 x+6 x$ and then factor by grouping. Here are the steps...

$$
\begin{aligned}
3 x^{2}+8 x+4 & =3 x^{2}+2 x+6 x+4 \\
& =x(3 x+2)+2(3 x+2) \\
& =[x+2](3 x+2)
\end{aligned}
$$

Remark: Note that using the ac test and the table above, we can conclude that any polynomial of the form $3 x^{2}+b x+4$ will factor if, and only if, $b=13,8,7,-13,-8$ or -7 . For any other value of $b$, the polynomial $3 x^{2}+b x+4$ will not factor over the integers!

Here are some more examples:

Example 2: Factor, if possiible: $6 x^{2}-17 x+5$.
Solution: First, note that $a c=30$ and $b=-17$. So we're looking for $p$ and $q$ with $p q=30$ and $p+q=-17$.
Here's the table of all pairs of factors of 30 , and their sums:

| p | q | $\mathrm{p}+\mathrm{q}$ |
| ---: | ---: | ---: |
| 1 | 30 | 31 |
| 2 | 15 | 17 |
| 3 | 10 | 13 |
| 5 | 6 | 11 |
| -1 | -30 | -31 |
| -2 | -15 | -17 |
| -3 | -10 | -13 |
| -5 | -6 | -11 |

We see that $p=-2$ and $q=-15$ works, so the polynomial passes the ac test, and thus will factor. To finish the problem, we factor by grouping:

$$
\begin{aligned}
6 x^{2}-17 x+5 & =6 x^{2}-2 x-15 x+5 \\
& =2 x(3 x-1)-5(3 x-1) \\
& =[2 x-5](3 x-1)
\end{aligned}
$$

Example 4: Factor, if possiible: $2 x^{2}-13 x+20$.
Solution: First, note that $a c=40$ and $b=-13$. So we're looking for $p$ and $q$ with $p q=40$ and $p+q=-13$.
Here's the table of all factors of 40 and their sums:

| p | q | $\mathrm{p}+\mathrm{q}$ |
| ---: | :---: | ---: |
| 1 | 40 | 41 |
| 2 | 20 | 22 |
| 4 | 10 | 14 |
| 5 | 8 | 13 |
| -1 | -40 | -41 |
| -2 | -20 | -22 |
| -4 | -10 | -14 |
| -5 | -8 | -13 |

We see that $p=-5$ and $q=-8$ works, so the polynomial passes the ac test, and thus will factor. To finish the problem, we factor by grouping:

$$
\begin{aligned}
2 x^{2}-13 x+20 & =2 x^{2}-5 x-8 x+20 \\
& =x(2 x-5)-4(2 x-5) \\
& =[x-4](2 x-5)
\end{aligned}
$$

Example 3: Factor, if possiible: $10 x^{2}+5 x-4$.
Solution: First, note that $a c=-40$ and $b=5$. So we're looking for $p$ and $q$ with $p q=-40$ and $p+q=5$. Here's the table of all factors of -40 and their sums:

| p | q | $\mathrm{p}+\mathrm{q}$ |
| ---: | ---: | ---: |
| 1 | -40 | -39 |
| 2 | -20 | -18 |
| 4 | -10 | -6 |
| 5 | -8 | -3 |
| -1 | 40 | 39 |
| -2 | 20 | 18 |
| -4 | 10 | 6 |
| -5 | 8 | 3 |

We see that no appropriate pair of $p$ and $q$ exists here. That is, there is no pair of factors of -40 which add to 5 . Thus the polynomial $10 x^{2}+5 x-4$ fails the ac test, and it will not factor over the integers.

Example 5: Factor, if possiible: $9 x^{2}+3 x-4$.
Solution: First, note that $a c=-36$ and $b=3$. So we're looking for $p$ and $q$ with $p q=-36$ and $p+q=3$. Here's the table of all factors of -36 and their sums:

| p | q | $\mathrm{p}+\mathrm{q}$ |
| ---: | ---: | ---: |
| 1 | -36 | -35 |
| 2 | -18 | -16 |
| 3 | -12 | -9 |
| 4 | -9 | -5 |
| 6 | -6 | 0 |
| -1 | 36 | 35 |
| -2 | 18 | 16 |
| -3 | 12 | 9 |
| -4 | 9 | 5 |

We see that no appropriate pair of $p$ and $q$ exists here. That is, there is no pair of factors of -36 which add to 3 . Thus the polynomial $9 x^{2}+3 x-4$ fails the ac test, and it will not factor over the integers.

## Practice Exercises

1) Use the ac test and ac method to factor the following polynomials, if possible.
a) $4 x^{2}+16 x+7$
b) $6 x^{2}-25 x+4$
c) $4 x^{2}+29 x+6$
d) $11 x^{2}+20 x-4$
2) For which values of the coefficient $b$ will the polynomial $5 x^{2}+b x-8$ factor over the integers?
