

Project # 1: Calculation of the Solar Interior Model

Obtaining the complete solution of the stellar structure equations (namely, ρ , T , M , and L at each point r inside the star), from scratch is somewhat tedious and tricky, but a partial solution can be obtained quite easily.

A partial solar model is given in the attached sheet, where the temperature T , the density ρ , and hydrogen abundance X are listed as function of the radius, expressed in terms of the radius of the Sun. Our aim is to fill in the columns for M_r , the mass contained within the sphere of radius r inside the star, and for L_r , the heat flux going through the sphere of radius r . Once we have done that, we will have ρ , T , M , L , and X at each point r , i.e., we will have determined the structure of the star. The results at the surface should approximate closely the values of the total mass and total luminosity for the Sun which is 1.99×10^{33} g and 3.9×10^{33} erg / s.

Since ρ and T are given, we only need to use the second and fourth stellar structure equations, namely,

$$\begin{aligned}\Delta M_r &= 4\pi r^2 \rho \Delta r \\ \Delta L_r &= 4\pi r^2 \rho \varepsilon \Delta r\end{aligned}$$

where ε is the energy generation rate, which in case of the p-p chain, the most important mechanism for the Sun, is given by:

$$\varepsilon = 2.8 \times 10^{-33} X^2 \rho T^{4.5} \quad (1)$$

where X is the fraction in mass of hydrogen.

Procedure:

If the functions M_r and L_r were linear functions of r , one could easily compute M_r and L_r at point r_{i+1} from M_r and L_r at point r_i , since $[\Delta M_r / \Delta r]$ and $[\Delta L_r / \Delta r]$ would give us the slopes of straight lines:

$$\begin{aligned}M_r(r_{i+1}) &= M_r(r_i) + \left(\frac{\Delta M_r}{\Delta r}\right)_i \Delta r \\ L_r(r_{i+1}) &= L_r(r_i) + \left(\frac{\Delta L_r}{\Delta r}\right)_i \Delta r .\end{aligned}$$

However, neither M_r nor L_r is linear. We can use a numerical technique called trapezoid rule to estimate $(M_r)_{i+1} = M_r(r_{i+1})$ and $(L_r)_{i+1} = L_r(r_{i+1})$ given $(M_r)_i = M_r(r_i)$ and $(L_r)_i = L_r(r_i)$, which is based in the same principle as for linear functions. In this technique $(M_r)_{i+1}$ and $(L_r)_{i+1}$ are given by:

$$(M_r)_{i+1} = (M_r)_i + \left\langle \frac{\Delta M_r}{\Delta r} \right\rangle \Delta r$$

$$(L_r)_{i+1} = (L_r)_i + \left\langle \frac{\Delta L_r}{\Delta r} \right\rangle \Delta r$$

where $\langle \dots \rangle$ is a mean slope, given by the average of the slopes at points r_{i+1} and r_i :

$$\left\langle \frac{\Delta M_r}{\Delta r} \right\rangle = \frac{1}{2} \left[\left(\frac{\Delta M_r}{\Delta r} \right)_i + \left(\frac{\Delta M_r}{\Delta r} \right)_{i+1} \right]$$

$$\left\langle \frac{\Delta L_r}{\Delta r} \right\rangle = \frac{1}{2} \left[\left(\frac{\Delta L_r}{\Delta r} \right)_i + \left(\frac{\Delta L_r}{\Delta r} \right)_{i+1} \right]$$

With these:

$$(M_r)_{i+1} = (M_r)_i + \frac{1}{2} \left[\left(\frac{\Delta M_r}{\Delta r} \right)_i + \left(\frac{\Delta M_r}{\Delta r} \right)_{i+1} \right] \Delta r$$

$$(L_r)_{i+1} = (L_r)_i + \frac{1}{2} \left[\left(\frac{\Delta L_r}{\Delta r} \right)_i + \left(\frac{\Delta L_r}{\Delta r} \right)_{i+1} \right] \Delta r$$
(2)

where

$$\left(\frac{\Delta M_r}{\Delta r} \right)_i = 4\pi r_i^2 \rho_i$$

$$\left(\frac{\Delta L_r}{\Delta r} \right)_i = 4\pi r_i^2 \varepsilon_i \rho_i$$

; $\Delta r = r_{i+1} - r_i$.

(3)

In computation, it is convenient to change the variables into dimensionless variables. We can introduce the following dimensionless variables: $m = (M_r/M_\odot)$, $l = (L_r/L_\odot)$, and $x = (r/R_\odot)$, in which case, equation (3) can be written as:

$$\left(\frac{\Delta m}{\Delta x} \right)_i = Ax_i^2 \rho_i$$

$$\left(\frac{\Delta l}{\Delta x} \right)_i = Bx_i^2 \rho_i \varepsilon_i$$
(4)

where $A = \text{constant} = 2.13011$, and $B = \text{constant} = 1.08636$. Equation (2) can be rewritten as:

$$m_{i+1} = m_i + \frac{1}{2} \left[\left(\frac{\Delta m}{\Delta x} \right)_{i+1} + \left(\frac{\Delta m}{\Delta x} \right)_i \right] \Delta x$$

$$l_{i+1} = l_i + \frac{1}{2} \left[\left(\frac{\Delta l}{\Delta x} \right)_{i+1} + \left(\frac{\Delta l}{\Delta x} \right)_i \right] \Delta x$$
(5)

where $\Delta x = x_{i+1} - x_i$.

We can follow the following procedure to generate a model of the stellar structure for the Sun:

1. Initial values are: $m_i = l_i = x_i = 0$ (index $i = 1$). These are the starting points.
2. Calculate ϵ_i at each point in the model, using equation (1).
3. Calculate $(\Delta m/\Delta x)_i$ and $(\Delta l/\Delta x)_i$ at each point, by using equation (4).
4. Calculate m_{i+1} and l_{i+1} at point x_{i+1} using equation (5), the values of the slopes at points x_{i+1} and x_i from equation (4), and the values of m_i and l_i from the previous step.

Application:

1. Calculate value m and l at each interior point in your model. Remember that at the surface ($r = R_\odot$ or $x = 1$) your results should approach closely the total mass of the Sun and the total luminosity of the Sun, if your model is right. You are checking your model against observations.
2. Once you have calculated (M_r/M_\odot) and (L_r/L_\odot) at each point (r/R_\odot) inside the star, plot graphs of them as function of (r/R_\odot) .
 - a). The "core" of the star may be defined as the region within which 90% of the energy is generated. This corresponds to the point where $(L_r/L_\odot) \sim 0.9$. From your plots, what are the mass and the radius of the core, compared to those of the total star? Compare with the radius of Jupiter and the Earth. Why is the energy generation concentrated in such a small region?
 - b). If the Sun was at a uniform density, the mass contained in a sphere of half the total radius, $(r/R_\odot) = 0.5$, would be $1/8 M_\odot$. Prove it.
 - c). What is the fraction of the total mass contained within a sphere of radius $(r/R_\odot) = 0.5$ from your plots? Would uniform density throughout the Sun a good approximation?
 - d). Draw a schematic Sun, indicating the different regions you have calculated: core, envelope. Indicate also the region where most of the mass is concentrated.
 - e). Calculate the effective temperature of your Sun. Compare to the accepted value $T_{eff} = 5,770$ K.

A Model of the Sun

i	$x = r/R_{\odot}$	ρ g/cm⁻³	T x10⁶K	X	ϵ	$\Delta m/\Delta x$	$\Delta l/\Delta x$	m	l
1	0.0	134.28	14.6	0.494					
2	0.04	121.34	14.3	0.520					
3	0.08	98.86	13.3	0.571					
4	0.12	76.62	12.0	0.643					
5	0.16	52.60	10.6	0.694					
6	0.20	36.39	9.35	0.723					
7	0.24	24.49	8.17	0.733					
8	0.28	16.03	7.11	0.741					
9	0.32	10.26	6.21	0.744					
10	0.36	6.55	5.42	0.744					
11	0.40	4.13	4.74	0.744					
12	0.44	2.60	4.16	0.744					
13	0.48	1.64	3.66	0.744					
14	0.52	1.03	3.21	0.744					
15	0.56	0.65	2.83	0.744					
16	0.60	0.40	2.49	0.744					
17	0.64	0.25	2.19	0.744					
18	0.68	0.16	1.93	0.744					
19	0.72	0.0971	1.69	0.744					
20	0.76	0.0590	1.47	0.744					
21	0.80	0.0354	1.27	0.744					
22	0.84	0.0212	1.04	0.744					
23	0.88	0.0129	0.741	0.744					
24	0.92	0.0066	0.473	0.744					
25	0.96	0.0022	0.473	0.744					
26	1.00			0.744					