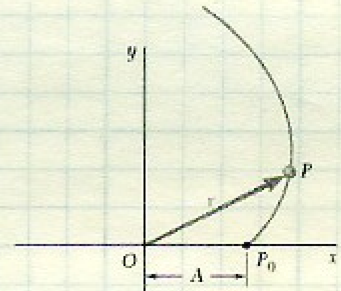
Set #3 – Curvilinear Motion

1. The motion of a particle is defined by the position vector $\mathbf{r} = A(\cos t + t \sin t)\mathbf{i} + A(\sin t - t \cos t)\,\mathbf{j}$, where t is expressed in seconds. Determine the values of t for which the position vector and the acceleration are

a) perpendicular,

b) parallel.



Given: $\vec{r} = A(\cos t + t \sin t)\hat{\imath} + A(\sin t - t\cos t)\hat{\jmath}$

t is in seconds

Find: $t \ni \vec{r} \perp \vec{a}$

$t \ni \vec{r} \parallel \vec{a}$

Solution:

$$\vec{v} = \frac{d\vec{r}}{dt} = A(-\sin t + t\cos t + \sin t)\hat{\imath} + A(\cos t + t\sin t - \cos t)$$

$$\vec{v} = (A t \cos t)\hat{\imath} + (A t \sin t)\hat{\jmath}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = At(-\sin t\,\hat{\imath} + \cos t\,\hat{\jmath}) + A(\cos t\,\hat{\imath} + \sin t\,\hat{\jmath})$$

$$\vec{a} = A(\cos t - t\sin t)\hat{\imath} + A(t\cos t + \sin t)\hat{\jmath}$$

For $\vec{r} \perp \vec{a}$, $\vec{r}\cdot\vec{a} = 0$

$$= \left[A(\cos t + t\sin t)\hat{\imath} + A(\sin t - t\cos t)\hat{\jmath}\right] \cdot \left[A(\cos t - t\sin t)\hat{\imath} + A(t\cos t + \sin t)\hat{\jmath}\right]$$

$$= A^2(\cos t + t\sin t)(\cos t - t\sin t) + A^2(\sin t - t\cos t)(t\cos t + \sin t)$$

$$= \cos^2 t - t\cos t\sin t + t\sin t\cos t - t^2\sin^2 t + t\sin t\cos t + \sin^2 t - t^2\cos^2 t - t\cos t\sin t$$

$$= 1 - t^2\sin^2 t - t^2\cos^2 t = 0$$

$\rightarrow 1 - t^2 = 0$

$1 = t^2$

$t = \pm 1$

$\underline{\underline{t = 1\,\text{sec}}}$ when $\vec{r} \perp \vec{a}$

Set #3

1. continued

For $\vec{r} \parallel \vec{a}$, $\vec{r} \times \vec{a} = \vec{0}$

$$\vec{r} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t + t\sin t & \sin t - t\cos t & 0 \\ \cos t - t\sin t & t\cos t + \sin t & 0 \end{vmatrix} = \vec{0}$$

$$= (\cos t + t\sin t)(t\cos t + \sin t) - (\cos t - t\sin t)(\sin t - t\cos t) = 0$$

$$= t + t\cos^2 t + t\sin^2 t = 0$$

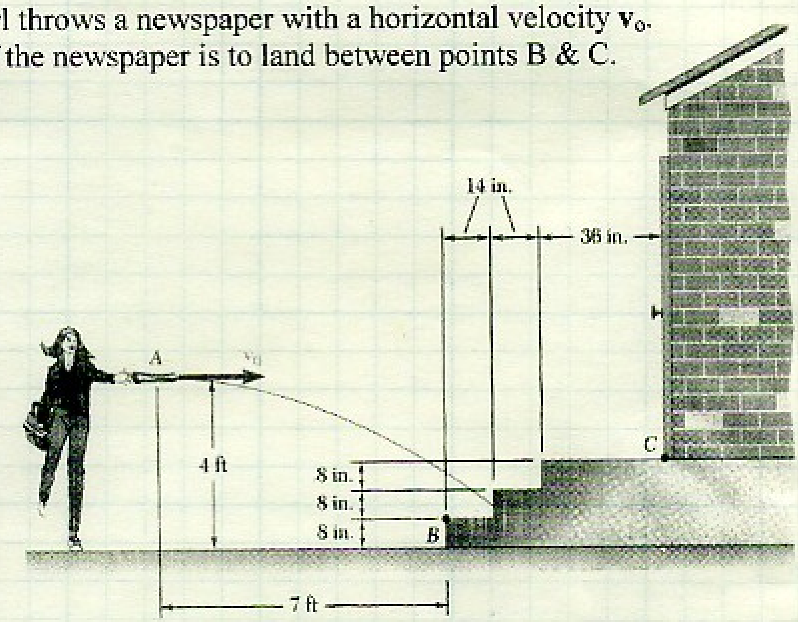
$$\longrightarrow t + t = 0$$

$$2t = 0$$

$\underline{\underline{t = 0 \text{ sec}}}$ is when $\vec{r} \parallel \vec{a}$

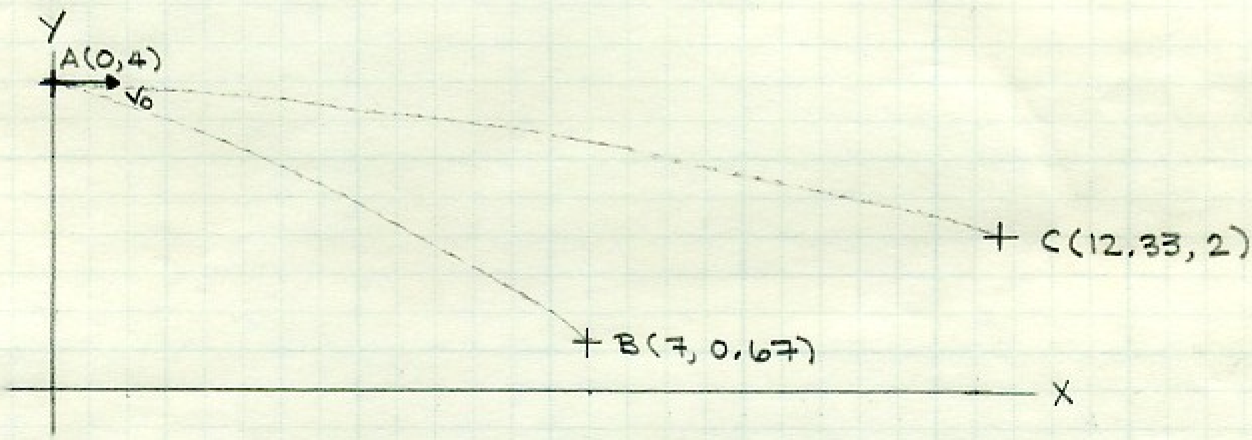
Set #3

2. While delivering newspapers, a girl throws a newspaper with a horizontal velocity $\mathbf{v}_o$. Determine the range of values of $\mathbf{v}_o$ if the newspaper is to land between points B & C.



Given:

All dimensions are in feet. → $g = 32.2\ ft/s^2$



Find: V_o necessary to reach B = V_o min
V_o necessary to reach C = V_o max.

Solution:

Projectile Motion:

$X_f = X_o + V_{ox} t$ $\qquad Y_f = Y_o + V_{oy} t - \frac{1}{2} g t^2$

$X_f = 0 + V_{ox} t$ $\qquad Y_f = 4 + 0 - \frac{1}{2}(32.2) t^2$

$X_f = V_{ox} t$

$\rightarrow t = \frac{X_f}{V_{ox}}$ — (1) $\qquad \rightarrow Y_f = 4 - 16.1 t^2$ — (2)

Substitute (1) into (2)

Set #3

2. (continued)

Substitute ① into ②

$$y_f = 4 - 16.1\left(\frac{x_f}{v_{ox}}\right)^2$$

$$y_f = 4 - \frac{16.1\,x_f^2}{v_{ox}^2}$$

$$\frac{16.1\,x_f^2}{v_{ox}^2} = 4 - y$$

$$\rightarrow v_{ox} = \sqrt{\frac{16.1\ x_f^2}{4-y}} \quad *$$

for (x, y) at B: $v_o = \sqrt{\frac{16.1\,(7)^2}{4-0.67}} = \underline{\underline{15.38\ ft/s = v_o\ min}}$

for (x, y) at C: $v_o = \sqrt{\frac{16.1\,(12.33)^2}{4-2}} = \underline{\underline{35\ ft/s = v_o\ max}}$

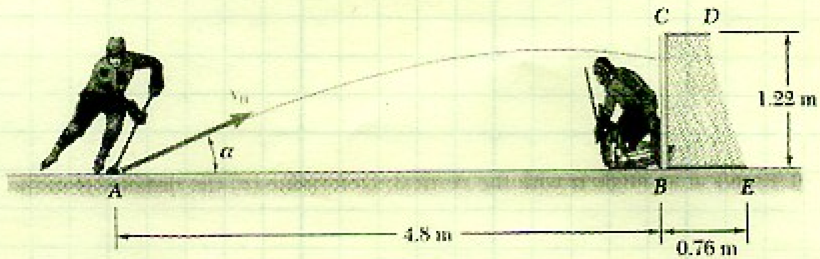
$$\underline{\underline{15.38\ ft/s \le v_o \le 35\ ft/s}}$$

Set #3 – Curvilinear Motion

3. The initial velocity v_0 of a hockey puck is 170 km/h. Determine

a) the largest value (less than 45°) of the angle α for which the puck will enter the net,

b) the corresponding time required for the puck to reach the net.



SOLUTION

Horizontal motion: $$x = (v_0 \cos\alpha)t \qquad \text{or} \qquad t = \frac{x}{v_0 \cos\alpha}$$

Vertical motion: $$y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

$$= x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$$

$$= x\tan\alpha - \frac{gx^2}{2v_0^2}\left(1+\tan^2\alpha\right)$$

$$\tan^2\alpha - \frac{2v_0^2}{gx}\tan\alpha + \left(1 + \frac{2v_0^2 y}{gx^2}\right) = 0$$

Data: $$v_0 = 170 \text{ km/h} = 47.222 \text{ m/s}, \qquad x = 4.8 \text{ m at point } C,$$

$$y = 1.22 \text{ m at point } C.$$

$$\frac{2v_0^2}{gx} = \frac{(2)(47.222)}{(9.81)(4.8)} = 94.712$$

$$\frac{2v_0^2 y}{gx^2} = \frac{(94.712)(1.22)}{4.8} = 24.073$$

(*a*) $$\tan^2\alpha - 94.712\alpha + 25.073 = 0$$

$$\tan\alpha = 0.26547 \qquad \text{and} \qquad 94.45$$

$$\alpha = 14.869° \qquad \text{or} \qquad 89.4°$$

$\alpha = 14.9°$ ◄

(*b*) $$t = \frac{x}{v_0\cos\alpha} = \frac{4.8}{(47.222)\cos 14.869°}$$

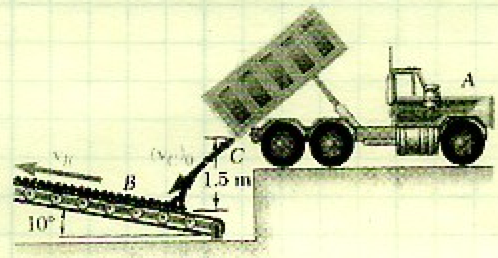
$t = 0.1052$ s ◄

Set #3 – Curvilinear Motion

4. Coal discharged from a dump truck with an initial velocity $(\mathbf{v}_C)_o = 1.8$ m/s at 50° ↙ falls onto conveyor belt B. Determine the required velocity $\mathbf{v}_B$ of the belt if the relative velocity with which the coal hits the belt is to be

a) vertical,

b) as small as possible.



SOLUTION

First determine the velocity $\mathbf{v}_C$ of the coal at the point where the coal impacts on the belt.

Horizontal motion: $(v_C)_x = [(v_C)_x]_0 = -1.8\cos 50°$

$= -1.1570$ m/s

Vertical motion: $(v_C)_y^2 = [(v_C)_y]_0^2 - 2g(y - y_0)$

$= (1.8\sin 50°)^2 - (2)(9.81)(-1.5)$

$= 31.331 \text{ m}^2/\text{s}^2$

$(v_C)_y = -5.5974$ m/s

$$\tan\beta = \frac{-5.5974}{-1.1570} = 4.8379, \quad \beta = 78.32°$$

$$v_C^2 = (v_C)_x^2 + (v_C)_y^2 = 32.669 \text{ m}^2/\text{s}^2$$

$$v_C = 5.7156 \text{ m/s}, \quad \mathbf{v}_C = 5.7156 \text{ m/s} \nearrow 78.32°$$

or $\mathbf{v}_C = (-1.1570 \text{ m/s})\mathbf{i} + (-5.5974 \text{ m/s})\mathbf{j}$

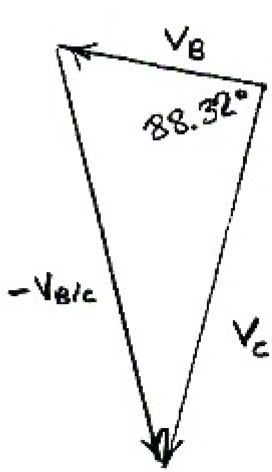
Velocity of the belt: $\mathbf{v}_B = v_B(-\cos 10°\mathbf{i} + \sin 10°\mathbf{j})$

Relative velocity: $\mathbf{v}_{C/B} = \mathbf{v}_C - \mathbf{v}_B = \mathbf{v}_C + (-\mathbf{v}_B)$

(a) $\mathbf{v}_{C/B}$ is vertical. $(v_{C/B})_x = 0$

$$(v_{C/B})_x = -1.1570 - v_B(-\cos 10°) = 0, \quad v_B = 1.175 \text{ m/s}$$

$\mathbf{v}_B = 1.175$ m/s ⦣ 10° ◀



(b) $v_{C/B}$ is minimum. Sketch the vector addition as shown.

$$v_{B/C}^2 = v_B^2 + v_C^2 - 2v_Bv_C\cos 88.32°$$

Set the derivative with respect to v_B equal to zero.

$$2v_B - 2v_C\cos 88.32° = 0$$

$v_B = v_C\cos 88.32° = 0.1676$ m/s $\quad$ $\mathbf{v}_B = 0.1676$ m/s ⦣ 10° ◀