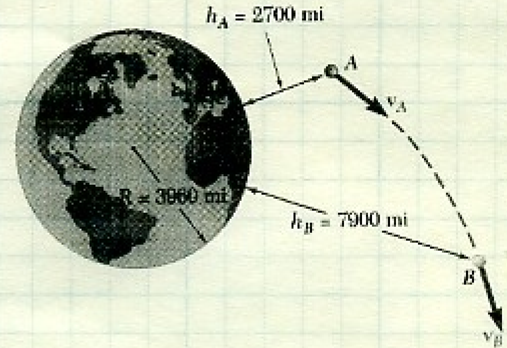
Set #9 – Applications to Space Mechanics

1. Knowing that the velocity of an experimental space probe fired from the earth has a magnitude $v_A = 20.2 \times 10^3$ mph at point A, determine the velocity of the probe as it passes through point B.



Given:
$$R_E = 3960 \text{ mi} = 2.0909 \times 10^7 \text{ ft}$$
$$v_A = 20.2 \times 10^3 \text{ mph} = 29{,}626.7 \text{ ft/s}$$
$$h_A = 2700 \text{ mi} = 1.4256 \times 10^7 \text{ ft}$$
$$r_A = R_E + h_A = 2.0909 \times 10^7 \text{ ft} + 1.4256 \times 10^7 \text{ ft} = 3.5165 \times 10^7 \text{ ft}$$
$$h_B = 7900 \text{ mi} = 4.1712 \times 10^7 \text{ ft}$$
$$r_B = R_E + h_B = 6.2621 \times 10^7 \text{ ft}$$

Find: v_B

Solution:
$$T_A + V_{g_A} = T_B + V_{g_B}$$
$$\tfrac{1}{2}\cancel{m}v_A^2 - \frac{GM\cancel{m}}{r_A} = \tfrac{1}{2}\cancel{m}v_B^2 - \frac{GM\cancel{m}}{r_B}$$
$$v_A^2 - \frac{2GM}{r_A} = v_B^2 - \frac{2GM}{r_B}$$
$$v_A^2 - \frac{2R_E^2 g}{r_A} = v_B^2 - \frac{2R_E^2 g}{r_B}$$
$$v_B^2 = v_A^2 + 2R_E^2 g\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$
$$v_B = \sqrt{v_A^2 + 2R_E^2 g\left(\frac{1}{r_B} - \frac{1}{r_A}\right)}$$
$$= \sqrt{(29{,}626.7)^2 + 2(2.0909 \times 10^7)^2(32.2)\left(\frac{1}{6.2621 \times 10^7} - \frac{1}{3.5165 \times 10^7}\right)}$$
$$= \underline{22949.9 \text{ ft/s}}$$
$$= \underline{\underline{15{,}647 \text{ mph}}}$$

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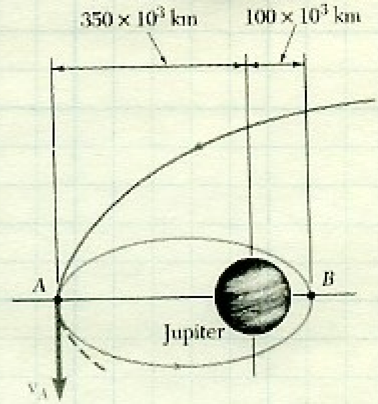
2. A spacecraft traveling along a parabolic path toward the planet Jupiter is expected to reach point A with a velocity $\mathbf{v}_A$ of magnitude 26.9 km/s. Its engines will then be fired to slow it down, placing it into an elliptic orbit which will bring it to within 100×10^3 km of Jupiter. Determine the decrease in speed Δv at point A which will place the spacecraft into the required orbit. The mass of Jupiter is 319 times the mass of the earth.



Given: $v_A = 26.9$ km/s
$= 2.69 \times 10^4$ m/s ↓
$m_J = 319\, M_E$

Find: Δv_A need to enter orbit.

Solution:

We need to find v_A such that the spacecraft enters the orbit passing through B.

$$r_A \cancel{m} v_A \sin\phi_A = r_B \cancel{m} v_B \sin\phi_B$$
$$r_A v_A \sin 90° = r_B v_B \sin 90°$$
$$r_A v_A = r_B v_B \quad (1)$$

$$T_A + V_{g_A} = T_B + V_{g_B}$$
$$\tfrac{1}{2} m v_A^2 - \frac{GMm}{r_A} = \tfrac{1}{2} m v_B^2 - \frac{GMm}{r_B}$$
$$v_A^2 = v_B^2 + 2GM\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

5:40
5:45

Using equation (1), substitute for v_B.

$$v_A^2 = \left(\frac{r_A v_A}{r_B}\right)^2 + 2GM\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$
$$v_A^2 = \frac{r_A^2 v_A^2}{r_B^2} + 2GM\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$
$$v_A^2 - \frac{r_A^2 v_A^2}{r_B^2} = 2GM\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$
$$v_A^2\left(1 - \frac{r_A^2}{r_B^2}\right) = 2GM\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$
$$v_A^2 = \frac{2GM\left(\frac{1}{r_A} - \frac{1}{r_B}\right)}{\left(1 - \frac{r_A^2}{r_B^2}\right)}$$

Set #9

2. continued

$$V_A^2 = \frac{2GM\left(\frac{1}{r_A} - \frac{1}{r_B}\right)}{\left(1 - \frac{r_A^2}{r_B^2}\right)}$$

$$V_A^2 = \frac{2(319 R_E^2 g)\left(\frac{1}{r_A} - \frac{1}{r_B}\right)}{\left(1 - \frac{r_A^2}{r_B^2}\right)}$$

$$V_A = \sqrt{\frac{2(319)(6370\times10^3)^2(9.81)\left(\frac{1}{350\times10^6} - \frac{1}{100\times10^6}\right)}{\left(1 - \frac{(350\times10^6)^2}{(100\times10^6)^2}\right)}}$$

$$= 12.7\times10^3 \text{ m/s}$$

So $\Delta V_A = 26.9 \text{ km/s} - 12.7 \text{ km/s}$

$\underline{\underline{\Delta V = 14.2 \text{ km/s}}}$

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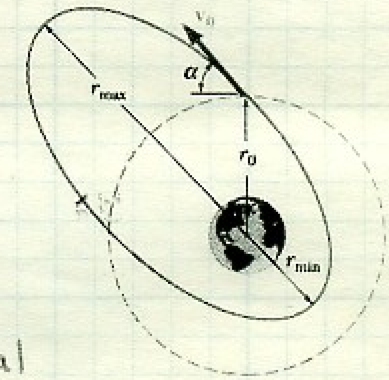
3. A satellite is projected into space with a velocity $\mathbf{v_0}$ at a distance r_0 from the center of the earth by the last stage of its launching rocket. The velocity $\mathbf{v_0}$ was designed to send the satellite into a circular orbit of radius r_0. However, owing to a malfunction of control, the satellite is not projected horizontally but at an angle α with the horizontal and, as a result, is propelled into an elliptic orbit. Determine the maximum and minimum values of the distance from the center of the earth to the satellite.



Given: Design → project V_0 horizontal for circular orbit of radius r_0

Malfunction → projected V_0 at angle α resulting in an elliptical orbit.

Find: r_{max}, r_{min}

Solution:

- We must first create a FBD of the satellite.

$$\Sigma F_n = ma_n$$

$$\longrightarrow \frac{GM_E m}{r_0^2} = \frac{mv_0^2}{r_0}$$

$$v_0 = \frac{GM}{r_0} \quad \text{—(1)}$$

- We can use the Conservation of Angular Momentum.

Recall: V_{min} pertains to the point r_{max}.
V_{max} pertains to the point r_{min}.

$$r_0 \cancel{m} V_0 \,(\sin 90 - \alpha) = r_{max} \cancel{m} V_{min} = r_{min} \cancel{m} V_{max}$$

$$r_0 V_0 \cos\alpha = r_{max} V_{min} = r_{min} V_{max}$$

Set #9

3. continued

We get

$$r_o V_o \cos\alpha = r_{max} V_{min}$$

$$\rightarrow V_{min} = \frac{r_o V_o \cos\alpha}{r_{max}} \quad \text{—(2)}$$

$$r_o V_o \cos\alpha = r_{min} V_{max}$$

$$\rightarrow V_{max} = \frac{r_o V_o \cos\alpha}{r_{min}} \quad \text{—(3)}$$

• Using the Conservation of Mechanical Energy,

$$T_o + V_o = T_A + V_A = T_B + V_B$$

$$\frac{1}{2} \cancel{m} V_o^2 - \frac{GM\cancel{m}}{r_o} = \frac{1}{2} \cancel{m} V_{min}^2 - \frac{GM\cancel{m}}{r_{max}} = \frac{1}{2} \cancel{m} V_{max}^2 - \frac{GM\cancel{m}}{r_{min}}$$

$$V_o^2 - \frac{2GM}{r_o} = V_{min}^2 - \frac{2GM}{r_{max}} = V_{max}^2 - \frac{2GM}{r_{min}}$$

$$\longrightarrow V_o^2 - \frac{2GM}{r_o} = V_{min}^2 - \frac{2GM}{r_{max}} \quad \text{—(4)}$$

$$\longrightarrow V_o^2 - \frac{2GM}{r_o} = V_{max}^2 - \frac{2GM}{r_{max}} \quad \text{—(5)}$$

Note: (4) and (5) are actually the same equation. Because this is a quadratic equation, we obtain two solutions, r_{max} and r_{min}. Hence, we will refer to r_{max} / r_{min} a, simply, r.

• Plugging in equation (2) into (4),

$$V_o^2 - \frac{2GM}{r_o} = \left(\frac{r_o V_o \cos\alpha}{r_{max}}\right)^2 - \frac{2GM}{r_{max}}$$

$$V_o^2 - \frac{2GM}{r_o} = \frac{r_o^2 V_o^2 \cos^2\alpha}{r^2} - \frac{2GM}{r}$$

$$\rightarrow \frac{r_o^2 V_o^2 \cos^2\alpha}{r^2} - \frac{2GM}{r} - V_o^2 + \frac{2GM}{r_o} = 0 \quad \text{—(6)}$$

Set #9

3. continued

Substitute ① into ⑥.

$$\frac{r_0^2\left(\frac{GM}{r_0}\right)\cos^2\alpha}{r^2} - \frac{2GM}{r} - \left(\frac{GM}{r_0}\right) + \frac{2GM}{r_0} = 0$$

$$\frac{r_0\cos^2\alpha}{r^2} - \frac{2}{r} - \frac{1}{r_0} + \frac{2}{r_0} = 0$$

$$\frac{r_0\cos^2\alpha}{r^2} - \frac{2}{r} + \frac{1}{r_0} = 0$$

$$r_0^2\cos^2\alpha - 2r_0 r + r^2 = 0$$

$$r^2 - (2r_0)r + r_0^2\cos^2\alpha = 0$$

$$r = \frac{-(-2r_0) \pm \sqrt{(-2r_0)^2 - 4(1)(r_0^2\cos^2\alpha)}}{2(1)}$$

$$= \frac{2r_0 \pm \sqrt{4r_0^2 - 4r_0^2\cos^2\alpha}}{2}$$

$$= \frac{2r_0 \pm 2r_0\sqrt{1-\cos^2\alpha}}{2}$$

$$= r_0 \pm r_0\sqrt{\sin^2\alpha}$$

$$= \underline{\underline{r_0(1 \pm \sin\alpha)}}$$