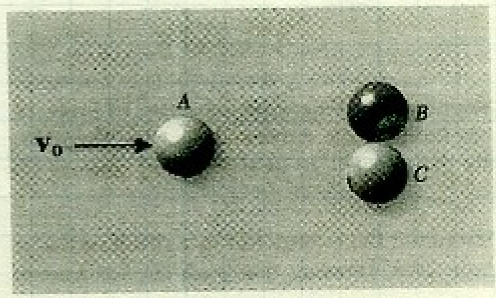
Set #13 – Systems of Particles

1. In a game of pool, ball A is moving with the velocity $\mathbf{v}_0 = v_0\mathbf{i}$ when it strikes balls B and C, which are at rest side by side. Assuming frictionless surfaces and perfectly elastic impact (i.e. conservation of energy), determine the final velocity of each ball, assuming that the path of A is

a) perfectly centered and that A strikes B and C simultaneously,
b) not perfectly centered and that A
c) strikes B slightly before it strikes C.

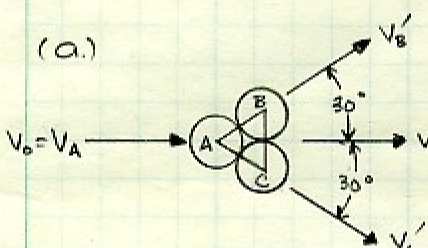


Given: $\vec{V}_A = V_A\hat{i}$
$\vec{V}_B = \vec{V}_C = 0$
Frictionless
Perfectly Elastic Collision

Find: $\vec{V}_A'$, $\vec{V}_B'$, $\vec{V}_C'$ if collision of A is
(a) perfectly centered (A strikes B and C simultaneously)
(b) not perfectly centered (A strike B before C.)

Solution:

(a.)



NOTE: After the collision, balls B and C move along the l.o.i. because there are no tangential forces. (V_B' and V_C' are as shown.)

By symmetry, V_A' has an x-component only. (The y-forces due to B and C will cancel.)

Also, through symmetry, $V_B' = V_C'$

Conservation of Momentum. (no external forces)

$$V_A = V_A' + 2V_B'\cos 30°$$
$$V_0 = V_A' + 2V_B'\cos 30°$$
$$V_A' = V_0 - 2V_B'\cos 30°$$
$$V_A' = V_0 - \sqrt{3}\,V_B' \quad \text{—(1)}$$

Set #13

1. continued

Conservation of Kinetic Energy.

$$\frac{1}{2} m v_0^2 = \frac{1}{2} m v_A'^2 + 2\left(\frac{1}{2} m v_B'^2\right)$$

$$v_0^2 = v_A'^2 + 2 v_B'^2 \quad -(2)$$

(1) into (2)

$$v_0^2 = (v_0 - \sqrt{3}\, v_B')^2 + 2 v_B'^2$$
$$v_0^2 = v_0^2 - 2\sqrt{3}\, v_0 v_B' + 3 v_B'^2 + 2 v_B'^2$$
$$v_0^2 = v_0^2 - 2\sqrt{3}\, v_0 v_B' + 5 v_B'^2$$
$$2\sqrt{3}\, v_0 v_B' = 5 v_B'^2$$
$$v_0 = \frac{5}{2\sqrt{3}} v_B'$$

$$v_B' = v_C' = \frac{2\sqrt{3}}{5} v_0$$

$$\underline{\underline{v_B' = v_C' = 0.693\, v_0}}$$

Using (1),

$$v_A' = v_0 - \sqrt{3}\left(\frac{2\sqrt{3}}{5}\right) v_0$$

$$v_A' = v_0 - \frac{6}{5} v_0$$

$$v_A' = -\frac{1}{5} v_0$$

$$\underline{\underline{v_A' = -0.2\, v_0}}$$

Set #13

1. continued

(b.) If A hits B first, • V_B' will remain the same. ($\measuredangle 30°$)
• V_A' will now have a y-component.

• Conservation of Momentum

$$V_0 \hat{i} = (V_{A_x}' \hat{i} + V_{A_y}' \hat{j}) + (V_B' \cos 30° \hat{i} + V_B' \sin 30° \hat{j})$$

x: $V_0 = V_{A_x}' + V_B' \cos 30°$

$\longrightarrow V_{Ax}' = V_0 - V_B' \cos 30°$ — 1

y: $0 = V_{Ay}' + V_B' \sin 30°$

$\longrightarrow V_{Ay}' = -V_B' \sin 30°$ — 2

• Conservation of Kinetic Energy

$$\frac{1}{2} m V_0^2 = \frac{1}{2} m (V_{Ax}'^2 + V_{Ay}'^2) + \frac{1}{2} m V_B'^2$$

$$V_0^2 = V_{Ax}'^2 + V_{Ay}'^2 + V_B'^2 \quad — 3$$

1 and 2 into 3

$$V_0^2 = (V_0 - V_B' \cos 30°)^2 + (-V_B' \sin 30°)^2 + V_B'^2$$

$$V_0^2 = V_0^2 - 2V_0V_B' \cos 30° + V_B'^2 \cos^2 30° + V_B'^2 \sin^2 30° + V_B'^2$$

$$2V_0V_B' \cos 30° = V_B'^2 \cos^2 30° + V_B'^2 \sin^2 30° + V_B'^2$$

$$2V_0 V_B' \frac{\sqrt{3}}{2} = V_B'^2 \frac{3}{4} + V_B'^2 \frac{1}{4} + V_B'^2$$

$$\sqrt{3} V_0 V_B' = 2V_B'^2$$

$$\sqrt{3} V_0 = 2V_B'$$

$$V_B' = \frac{\sqrt{3}}{2} V_0$$

$$\underline{\underline{V_B' = 0.866 \, V_0 \; \measuredangle 30°}}$$

Using 1 and 2,

$V_{Ax}' = V_0 - \frac{\sqrt{3}}{2} V_0 \frac{\sqrt{3}}{2}$

$V_{Ax}' = \frac{V_0}{4}$

$V_{Ax}' = .25 V_0$ — 4

$V_{Ay}' = -\frac{\sqrt{3}}{2} V_0 \frac{1}{2}$

$V_{Ay}' = -\frac{\sqrt{3}}{4} V_0$

$V_{Ay}' = -.433 V_0$ — 5

$$V_A' = \sqrt{V_{A_x}^2 + V_{Ay}^2}$$

$$= \frac{1}{2} V_0 \quad — 6$$

Set #13

1. continued

Now, A will hit C.

- v_c' will still be $\measuredangle 30^\circ$

- Conservation of Momentum in x-direction

$$v_{Ax}' = v_{Ax}'' + v_c'' \cos 30^\circ$$

$$\frac{v_o}{4} = v_{Ax}'' + \frac{\sqrt{3}}{2} v_c''$$

$$\longrightarrow v_{Ax}'' = \frac{v_o}{4} - \frac{\sqrt{3}}{2} v_c'' \quad — ⑦$$

- Cons. of Momentum in y-direction.

$$v_{Ay}' = v_{Ay}'' - v_c'' \sin 30^\circ$$

$$-\frac{\sqrt{3}}{4} v_o = v_{Ay}'' - v_c'' \sin 30^\circ$$

$$\longrightarrow v_{Ay}'' = \frac{v_c''}{2} - \frac{\sqrt{3}}{4} v_o \quad — ⑧$$

Using Pythagorean Theorem and ⑦ and ⑧, we get.

$$v_A''^2 = \frac{4 v_o^2}{16} - \frac{\sqrt{3}}{2} v_o v_c'' + v_c''^2 \quad — ⑨$$

- Cons. of Kinetic Energy

$$\frac{1}{2} m v_A'^2 = \frac{1}{2} m v_A''^2 + \frac{1}{2} m v_c''^2 \quad \text{(Plug-in ⑥ and ⑨)}$$

$$\frac{1}{4} v_o^2 = \frac{v_o^2}{4} - \frac{\sqrt{3}}{2} - v_o c'' + v_c''^2 + v_c''^2$$

$$\frac{\sqrt{3}}{2} v_o v_c'' = 2 v_c''^2$$

$$\frac{\sqrt{3}}{2} v_o = 2 v_c''$$

$$\longrightarrow v_c'' = \frac{\sqrt{3}}{4} v_o$$

$$\underline{\underline{= .433\, v_o}}$$

Set #13

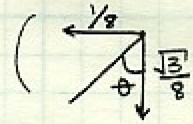
1. continued

$$V_{Ax}'' = \frac{V_o}{4} - \frac{\sqrt{3}}{2}\left(\frac{3}{4}V_o\right) = -\frac{V_o}{8}$$

$$V_{Ay}'' = \frac{V_c''}{2} - \frac{\sqrt{3}}{4}V_o = \frac{\sqrt{3}}{8}V_o - \frac{2\sqrt{3}}{8}V_o = -\frac{\sqrt{3}}{8}V_o$$

$$V_A'' = \sqrt{\frac{1}{64} + \frac{3}{64}}\;V_o$$

$$\underline{\underline{= 0.25\,V_o \qquad \theta = 30^\circ}}$$



Set #13 – Systems of Particles

2. A 360-kg space vehicle traveling with a velocity $\mathbf{v}_0 = (450 \text{ m/s})\mathbf{k}$ passes through the origin O. Explosive charges then separate the vehicle into three parts A, B, and C, with masses of 60 kg, 120 kg, and 180 kg respectively. Knowing that shortly thereafter the positions of the three parts are, respectively, A(72, 72, 648), B(180, 396, 972), and C(-144, -288, 576), where the coordinates are expresses in meters, that the velocity of B is $\mathbf{v}_B = (150 \text{ m/s})\mathbf{i} + (330 \text{ m/s})\,\mathbf{j} + (600\text{m/s})\,\mathbf{k}$, and that the x component of the velocity of C is -120 m/s, determine the velocity of part A.

Given: $m = 360$ kg , $\vec{r} = \langle 0, 0, 0\rangle$, $\vec{v}_0 = \langle 0, 0, 450\rangle$ m/s

$m_A = 60$ kg , $\vec{r}_A = \langle 72, 72, 648\rangle$ m ,

$m_B = 120$ kg , $\vec{r}_B = \langle 180, 396, 972\rangle$ m , $\vec{v}_B = \langle 150, 330, 600\rangle$ m/s

$m_C = 180$ kg , $\vec{r}_C = \langle -144, -288, 576\rangle$ m , $\vec{v}_C = \langle -120, v_{cy}, v_{cz}\rangle$ m/s

$\vec{r}_A = 72\langle 1, 1, 9\rangle$ m ,

$\vec{r}_B = 18\langle 10, 22, 54\rangle$ m , $\vec{v}_B = 30\langle 5, 11, 22\rangle$ m/s

$\vec{r}_C = 144\langle -1, -2, 4\rangle$ m , $\vec{v}_C = \langle -120, v_{cy}, v_{cz}\rangle$ m/s

Find: $\vec{v}_A$

Solution:

- No external forces
  → Conservation of Momentum ($\vec{L}_1 = \vec{L}_2$)

$$m\vec{v}_0 = m_A\vec{v}_A + m_B\vec{v}_B + m_c\vec{v}_c$$

$$360\langle 0, 0, 450\rangle = 60\langle v_{Ax}, v_{Ay}, v_{Az}\rangle + 120\langle 150, 330, 600\rangle + 180\langle -120, v_{cy}, v_{cz}\rangle$$

$$\hat{i}: \quad 0 = 60v_{Ax} + 18000 - 21600$$
$$\hat{j}: \quad 0 = 60v_{Ay} + 39600 + 180v_{cy}$$
$$\hat{k}: \quad 162{,}000 = 60v_{Az} + 79200 + 180v_{cz}$$

$$\hat{i}: \quad v_{Ax} = 60 \text{ m/s} \quad (1)$$
$$\hat{j}: \quad v_{Ay} = -660 - 3v_{cy} \quad (2)$$
$$\hat{k}: \quad v_{Az} = 1380 - 3v_{cz} \quad (3)$$

- No external forces
  → Conservation of Angular Momentum ($\vec{H}_{o_1} = \vec{H}_{o_2}$)

$$\vec{H}_o = \vec{r}_A \times m_A\vec{v}_A + \vec{r}_B \times m_B\vec{v}_B + \vec{r}_c \times m_c\vec{v}_c$$

$$0 = (60)(72)\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 9 \\ 60 & v_{Ay} & v_{Az} \end{vmatrix} + (120)(18)(30)\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10 & 22 & 54 \\ 5 & 11 & 22 \end{vmatrix} + (180)(144)\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & 4 \\ -120 & v_{cy} & v_{cz} \end{vmatrix}$$

Set #13

2. continued

$$0 = (60)(72)\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 9 \\ 60 & V_{Ay} & V_{Az} \end{vmatrix} + (120)(18)(30)\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10 & 22 & 54 \\ 5 & 11 & 22 \end{vmatrix} + (180)(144)\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & 4 \\ -120 & V_{cy} & V_{cz} \end{vmatrix}$$

$$= 4320\left[(V_{Az} - 9V_{Ay})\hat{i} - (V_{Az} - 540)\hat{j} + (V_{Ay} - 60)\hat{k}\right]$$
$$+ 64800\,(-110\hat{i} + 50\hat{j} + 0\hat{k})$$
$$+ 25920\left[(-2V_{cz} - 4V_{cy})\hat{i} - (-V_{cz} + 480)\hat{j} + (-V_{cy} - 240)\hat{k}\right]$$

$\hat{i}$: $0 = 4320\,V_{Az} - 38880\,V_{Ay} - 7128000 - 51840\,V_{cz} - 103680\,V_{cy}$
$= V_{Az} - 9\,V_{Ay} - 1650 - 12\,V_{cz} - 24\,V_{cy}$
$= -9\,V_{Ay} + V_{Az} - 24V_{cy} - 12\,V_{cz} - 1650$

$\longrightarrow 1650 = -9V_{Ay} + V_{Az} - 24\,V_{cy} - 12V_{cz}$ — 4

$\hat{j}$: $0 = -4320\,V_{Az} + 2332800 + 3240000 + 25920\,V_{cz} - 12441600$
$= -V_{Az} + 540 + 750 + 6\,V_{cz} - 2880$
$= -V_{Az} + 6V_{cz} - 1590$

$\longrightarrow 1590 = -V_{Az} + 6V_{cz}$ — 5

$\hat{k}$: $0 = 4320\,V_{Ay} - 259200 + 0 - 25920\,V_{cy} - 6220800$
$= V_{Ay} - 60 - 6\,V_{cy} - 1440$
$= V_{Ay} - 6V_{cy} - 1500$

$\longrightarrow 1500 = V_{Ay} - 6V_{cy}$ — 6

Now, using the 6 equations, we are able to solve for all the remaining unknowns, V_{Ay}, V_{Az}, V_{cy}, V_{cz}.

Set #13

2. continued

① $V_{Ax} = 60 \text{ m/s}$

② $V_{Ay} = -660 - 3V_{cy}$

③ $V_{Az} = 1380 - 3V_{cz}$

④ $1650 = -9V_{Ay} + V_{Az} - 24V_{cy} - 12V_{cz}$

⑤ $1590 = -V_{Az} + 6V_{cz}$

⑥ $1500 = V_{Ay} - 6V_{cy}$

② into ⑥:
$$1500 = (-660 - 3V_{cy}) - 6V_{cy}$$
$$1500 = -660 - 3V_{cy} - 6V_{cy}$$
$$2160 = -9V_{cy}$$
$$\longrightarrow V_{cy} = -240 \text{ m/s}$$

②: $V_{Ay} = -660 - 3(-240)$

$V_{Ay} = 60 \text{ m/s}$

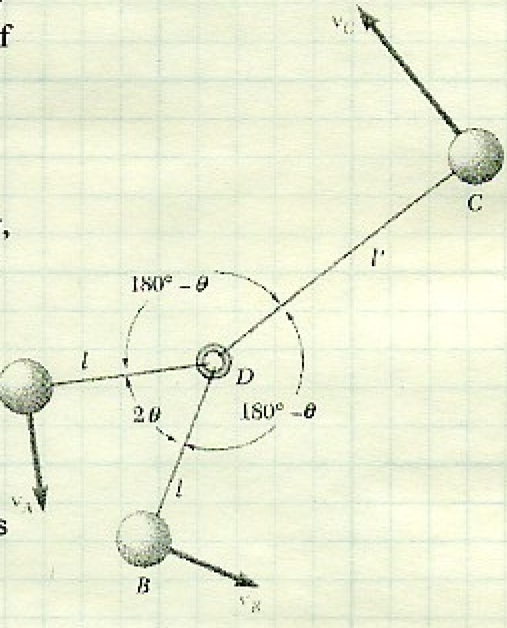
③ into ⑤:
$$1590 = -(1380 - 3V_{cz}) + 6V_{cz}$$
$$1590 = -1380 + 3V_{cz} + 6V_{cz}$$
$$2970 = 9V_{cz}$$
$$\longrightarrow V_{cz} = 330 \text{ m/s}$$

⑤
$$1590 = -V_{Az} + 6(330)$$
$$1590 = -V_{Az} + 1980$$
$$-390 = -V_{Az}$$
$$\longrightarrow V_{Az} = 390 \text{ m/s}$$

$$\underline{\underline{\vec{V}_A = \langle 60, 60, 390 \rangle \text{ m/s}}}$$

Set #13 – System of Particles

3. Three identical spheres A, B, and C, which can slide freely on a frictionless horizontal surface, are connected by means of inextensible, inelastic cords to a small ring D located at the mass center of the three spheres (l' = 2l cos θ). The spheres are rotating initially about rind D, which is at rest, at speeds proportional to their distances from D. We denote by v_o the original speed of A and B and assume that θ = 30°. Suddenly, cord Cd breaks, causing sphere C to slide away. Considering the motion of spheres A and B of ring D after the other two cords have again become taut, determine



a) the speed of ring D,
b) the relative speed at which spheres A and B rotate about D,
c) the percent of energy of the original system which is dissipated when cord Ad and BD again become taut.

Given: identical spheres
Frictionless horizontal surface
inextensible, inelastic cords
D is mass center of A, B, C
speed $\propto$ to distance from D
$v_{D_o} = 0$
CD breaks, C slides away
$l' = 2l \cos\theta$
$\theta = 30°$

Find: (a) $\vec{v}_D$
(b) $\vec{v}_{A/D}$, $\vec{v}_{B/D}$
(c.) % $\mathcal{E}$ dissipated

Solution:
(a)
• Using $m\bar{v} = \Sigma m_i v_i$ in $\hat{j}$-direction

$v_{D_o} = 0$

$\longrightarrow 0 = m v_{C_o} - 2 m v_o \cos\theta$

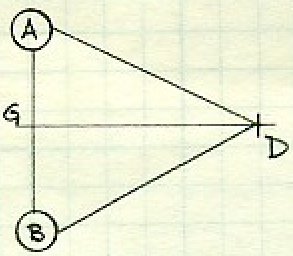
$v_{C_o} = 2 v_o \cos\theta$

Set #13

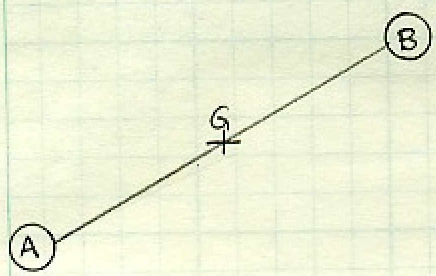
3. continued

- when the cord breaks:

There are no forces in the plane of motion.
$\rightarrow mv$ is conserved.



V_G immediately after break $= \dfrac{-mv_0\cos\theta\,\hat{j} - mv_0\cos\theta\,\hat{j}}{2m}$
$= -v_0\cos\theta\,\hat{j}$



V_G after string is taut $= -v_0\cos\theta\,\hat{j}$
$= \vec{V}_D$

$\vec{V}_D = -v_0\cos\theta\,\hat{j}$
$\underline{\vec{V}_D = -0.866\,v_0\,\hat{j}}$

- Angular momentum is also conserved because there are no forces in the plane of motion

$\vec{H}_G$ immediately after break $= 2(l\sin\theta)(mv_0\sin\theta)$
$\underline{\underline{= 2\,lm\,v_0\sin^2\theta}}$

$\vec{H}_G$ string taut $= H'_G$ string taut $= lm\,V_{A/G} + lm\,V_{B/G}$

$2lmv_0\sin^2\theta = 2\,lm\,V_{A/G}$

$\rightarrow V_{A/G} = v_0\sin^2\theta$

$\underline{\underline{V_{A/G} = 0.25\,V_0}}$

Set #13

3. Continued

(c.) $T_1 = 2(\frac{1}{2} m v_o^2) + \frac{1}{2} m (2 V_o \cos\theta)^2$
$= m v_o^2 + 1.5 \, m v_o^2$
$= 2.5 \, m v_o^2$

$T_2 = [\frac{1}{2} M \bar{v}^2 + \Sigma \frac{1}{2} m_i v_i'^2] + \frac{1}{2} m v_c^2$
$= \frac{1}{2}(2m)(.866 \, V_o)^2 + 2(\frac{1}{2} m)(.25 \, V_o)^2 + 1.5 \, m v_o^2$
$= \frac{3}{4} m v_o^2 + \frac{1}{16} m v_o^2 + 1.5 \, m v_o^2$
$= \frac{13}{16} m v_o^2 + \frac{3}{2} m v_o^2$
$= \frac{37}{16} m v_o^2$

$\% E \text{ dissipated} = \left(\frac{2.5 - 2.3125}{2.5}\right) 100\%$
$= \underline{\underline{7.5\%}}$